GENERALIZATION OF THE GROUPS OF GENUS ZERO*

BY

G. A. MILLER

One of the most important systems of groups is that which is composed of all the groups which may be defined by the orders of two generators (s_1, s_2) and the order of their product. It is known that these three orders determine a finite group only when one of them is unity, when two of them are equal to 2, or when one is 2 while the other two are one of the following three pairs of numbers: $3, 3$; $3, 4$; $3, 5$.† The first of these sets of orders define a cyclic group and may be regarded as trivial. When the order of two among the three operators $s_1, s_2, s_1 s_2$ is equal to 2, the group $\{s_1, s_2\}$, generated by s_1, s_2, is the dihedral rotation group whose order is twice that of the third operator. This useful category has recently been generalized under the heading, *The groups generated by two operators which have a common square*.‡ This generalized category could also be defined as the set of groups generated by two operators such that the square of one of them is equal to the square of their product.

The object of the present paper is to complete this kind of generalization for the dihedral rotation group and to extend it to the other groups of genus zero. The resulting groups have a two-fold interest in view of their close contact with the important system of groups of genus zero and their elementary structures. It is believed that a complete list of these groups will prove very useful in many investigations. Our object is to study all the groups which result when one of the three conditions which may be used to define a group of genus zero is preserved while the other two are replaced by a single one of an elementary type. For instance, the dihedral rotation groups are defined by the conditions §

$$s_1^2 = 1, \qquad s_2^2 = 1, \qquad (s_1 s_2^{-1})^n = 1.$$

In the generalization mentioned above these conditions are replaced by

* Presented to the Society at the New Haven meeting September 3, 1906. Received for publication July 11, 1906.

† American Journal of Mathematics, vol. 24 (1902), p. 96.

‡ Archiv der Mathematik und Physik, vol. 9 (1905), p. 6.

§ Throughout the paper care has been taken to insure that, when one of the given generational relations of a group is $s^n = 1$, the operator s shall be of order n.

$$s_1^2 = s_2^2, \qquad (s_1 s_2^{-1})^n = 1.$$

A similar generalization is given by the equations

$$s_1^2 = s_2^n, \qquad (s_1 s_2^{-1})^2 = 1 \qquad (n > 2).$$

We proceed to consider the groups whose generators satisfy the last equation. From $(s_1 s_2^{-1})^2 = 1$ it results that $s_1 s_2^{-1} s_1 = s_2$. Hence $s_1^{-1} s_2 s_1 = s_2^{n-1}$ and $s_1^{-1} s_2^n s_1 = s_2^{n^2-n}$ Since s_2^n is invariant under $\{s_1, s_2\}$ it is necessary that $n^2 - n \equiv n \pmod{m}$, m being the order of s_2. That is, m must divide $(n-1)^2 - 1$. Since $\{s_1, s_2\}$ is generated by s_2 and $s_1 s_2^{-1}$ and since the latter transforms the former into its $(n-1)$-th power, the order of $\{s_1, s_2\}$ is $2m$. These results may be expressed as follows: *If the two generators of a group satisfy the conditions* $s_1^2 = s_2^n$, $(s_1 s_2^{-1})^2 = 1$, $n > 2$, *it is non-abelian, involves a cyclic subgroup of half its order, and an additional operator of order* 2 *which transforms each operator of this cyclic subgroup into its* $(n-1)$*-th power.*

For $m = n$ this category of groups reduces to the dihedral rotation groups. It is clear that we may assume that $m \geqq n$; for, if m were less than n, the number n could be replaced by $n - km$, where k is arbitrary. The invariant operators of $\{s_1, s_2\}$ constitute a subgroup whose order is the highest common factor of m and $n-2$. With respect to this subgroup the quotient group of $\{s_1, s_2\}$ is a dihedral rotation group. If H represents any Sylow subgroup of odd order in $\{s_1, s_2\}$, then either s_1 is commutative with each operator of H or it transforms each of these operators into its inverse. In the former case the order of s_1 is divisible by that of H. In the latter case the orders of s_1 and H are relatively prime. The numbers n and m cannot be relatively prime since $\{s_1, s_2\}$ is non-cyclic. For the same reason s_1 must be of even order, as is also otherwise evident.

It may be added that if n is replaced by $n' = m - n$, the given equations become

$$t_1^2 = t_2^{n'}, \qquad (t_1 t_2)^2 = 1 \qquad (n' < m - 2).$$

Hence these equations define the same system of groups as those given above. Similar remarks apply to the system defined by the equations $t_1^2 = t_2^2$, $(t_1 t_2)^n = 1$. Since the dihedral rotation groups constitute an infinite system it is necessary that the generalization of the conditions satisfied by its generators are also satisfied by the generators of an infinite system of groups. The other groups of genus zero are defined as individuals, and the corresponding generalizations are satisfied by a very small number of groups.

§ 1. *Generalization of the tetrahedral group.*

It is known that any two operators of order 3 whose product is of order 2 generate the tetrahedral group. That is, this group is completely defined by two operators s_1, s_2 which satisfy either of the following sets of conditions:

$$s_1^3 = s_2^3 = 1, \ (s_1 s_2)^2 = 1; \qquad s_1^2 = s_2^3 = 1, \ (s_1 s_2)^3 = 1.$$

We proceed to consider the groups which satisfy the more general equations

$$s_1^3 = s_2^3, \ (s_1 s_2)^2 = 1; \qquad s_1^2 = s_2^3, \ (s_1 s_2)^3 = 1.$$

We begin with the first of these two cases; that is, we consider all the groups which are generated by two operators having a common cube and a product of order 2.

The three operators $s_1 s_2$, $s_2 s_1$, $s_1^{-1} s_2 s_1^2$ are conjugate under s_1 and the product of the last two is $s_2^2 s_1^2 = s_1 s_2 \cdot s_1^6$ since $s_1 s_2 = s_2^{-1} s_1^{-1}$. As $s_2 s_1$ transforms this product into its inverse and is commutative with s_1^6 the order of s_1 must divide 24. As it is a multiple of 3 it has one of the three valuees 24, 12, 6, 3. If this order is 24 the two operators of order two $s_2 s_1$, $s_1^{-1} s_2 s_1^2$ generate the octic group since $s_2^2 s_1^2$ is of order 4. This octic group and s_1^6 generate an invariant subgroup of order 16 since it is transformed into itself by s_1 and also by s_2. As this invariant subgroup contains the quaternion group as a characteristic subgroup, this quaternion group and s_1^8 generate the group of order 24 which does not involve a subgroup of order 12. Hence the given group of order 16 and s_1^8 generate the group of order 48 known as G_{50}.* As this involves s_1^6 and s_1^8 it also involves s_1^2. If we extend it by means of s_1^3 there results a group of order 96 which involves both s_1 and s_2 and hence is $\{s_1, s_2\}$. As s_1^3 is invariant under $\{s_1, s_2\}$ there is only one such group.† It may be represented as a substitution group of degree 32 by means of

$$s_1 = ac'ce'eg'ga' \cdot bm'nf'oj'hk'ld'mp'fjb'kn'do'ph'il',$$

$$s_2 = am'je'on'gk'pc'ml'ei'na'kj'co'lg'ip' \cdot bd'df'fh'hb'.$$

When s_1 is of order 12 the two operators $s_2 s_1$, $s_1^{-1} s_2 s_1^2$ are commutative since their product is of order 2. Hence $\{s_2 s_1, s_1^{-1} s_2 s_1^2\}$ is the four-group. This four-group and s_1^6 generate the group of order 8 which involves seven operators of order 2. This is invariant under $\{s_1, s_2\}$. Since the group of isomorphisms of this group of order 8 is known, it is known that s_1^4 and one of its subgroups of order 4 generate the tetrahedral group. Hence $\{s_1, s_2\}$ is the direct product of this tetrahedral group and s_1^3. When s_1 is of order 6 the four-group $\{s_2 s_1, s_1^{-1} s_2 s_1^2\}$ involves $s_1 s_2$, and together with s_1^2 generates the tetrahedral group. Hence $\{s_1, s_2\}$, in this case, is the direct product of the tetrahedral group and the group of order 2. These results may be expressed as follows:

*Quarterly Journal of Mathematics, vol. 30 (1898), p. 258.

†Bulletin of the American Mathematical Society, vol. 3 (1897), p. 218.

There are exactly four groups whose two generators s_1, s_2 *satisfy the relations* $s_1^3 = s_2^3$, $(s_1 s_2)^2 = 1$. *They are the tetrahedral group; the direct products of the tetrahedrl group and the cyclic group of order* 4 *or the group of order* 2; *and the group of order* 96 *considered above.*

The second generalization of the tetrahedral group relates to the groups whose two generators satisfy the equations

$$s_1^2 = s_2^3, \qquad (s_1 s_2)^3 = 1.$$

The three conjugates of s_1 under s_2 are

$$s_1, \qquad s_2^{-1} s_1 s_2, \qquad s_2^{-2} s_1 s_2^2.$$

As these conjugates have a common square we are concerned with a generalization of the dihedral rotation group. Multiplying the second into the inverse of the third, we get

$$s_2^{-1} s_1 s_2 \cdot s_2^{-2} s_1^{-1} s_2^2 = s_2^{-1} s_1 s_2^{-1} s_1 s_2^2 \cdot s_1^{-2} = s_1^5.$$

Since s_1^5 is transformed into its inverse by $s_2^{-1} s_1 s_2$ and s_1^2 is invariant under $\{s_1, s_2\}$ it follows that the order of s_1 divides 20. If the order of s_1 is 20 we consider the group generated by $s_1^{25} = t_1$, $s_2^{25} = t_2$. From the fact that $(s_1^{25} s_2^{25})^3 = s_1^{120}(s_1 s_2)^3 = 1$, it results that $t_1^2 = t_2^3$ and $(t_1 t_2)^3 = 1$. Hence $\{t_1, t_2^{-1} t_1 t_2, t_2^{-2} t_1 t_2^2\}$ is the quaternion group, which is invariant under $\{t_1, t_2\}$ since it includes t_1 and is transformed into itself by t_2. This quaternion group and t_2 generate the group of order 24 which does not contain a subgroup of order 12. Hence this is $\{t_1, t_2\}$, while $\{s_1, s_2\}$ is the direct product of $\{t_1, t_2\}$ and the cyclic group of order 5 generated by s_1^4.

When the order of s_1 is 10, it follows from the preceding paragraph that $\{s_1^{25}, s_2^{25}\}$ is the tetrahedral group and that $\{s_1, s_2\}$ is the direct product of this group and the cyclic group of order 5. These results may be expressed as follows: *There are exactly four groups whose two generators satisfy the conditions* $s_1^2 = s_2^3$, $(s_1 s_2)^3 = 1$. *They are the tetrahedral group, the group of order* 24 *which does not contain a subgroup of order* 12, *and the direct products of these groups and the cyclic group of order* 5. As only one of these groups occurs among the four given above we have seven distinct groups resulting from one or the other of the given generalizations of the defining equations of the tetrahedral group.

It may be of interest to note how a slight change in these generalizations may affect the system. Instead of the equations $s_1^3 = s_2^3$, $(s_1 s_2)^2 = 1$, we proceed to consider $s_1^3 = s_2^3$, $(s_1 s_2^{-1})^2 = 1$. The three conjugates $s_1 s_2^{-1}$, $s_2^{-1} s_1$, $s_1^{-1} s_2^{-1} s_1^2$ are commutative and constitute, together with the identity, the four-group since $s_2^{-2} s_1^2 = (s_1 s_2^{-1})^{-1} = s_1 s_2^{-1}$. This four-group is invariant under $\{s_1, s_2\}$ and has only the identity in common with the cyclic group generated by s_1^3. Hence the theorem:

Two operators s_1, s_2 *which satisfy the conditions* $s_1^3 = s_2^3$ *and* $(s_1 s_2^{-1})^2 = 1$ *generate the group which may be obtained by establishing a* $(4, h)$ *isomorphism between the tetrahedral group and a cyclic group of order* $3h$, *where* $3h$ *is the order of* s_1.*

Since there is one and only one such group for every value of h, a group is completely defined by the value of h and the given equations. In particular, when $h = 1$, $\{s_1, s_2\}$ is the tetrahedral group. When h is prime to 3, $\{s_1, s_2\}$ is the direct product of the tetrahedral group and the cyclic group of order h.†

Similarly, the second set of equations given at the beginning of this section may be replaced by $s_1^3 = s_2^2$ and $(s_1^{-1} s_2)^3 = 1$. The three conjugates of s_2 under s_1 are

$$s_2, \quad s_1^{-1} s_2 s_1, \quad s_1^{-2} s_2 s_1^2.$$

The product of the second into the inverse of the third is $s_1^{-1} s_2 s_1^{-1} s_1^2 = s_2^{-1}$. Since $s_1^{-1} s_2 s_1$ transforms s_2 into its inverse and is commutative with s_2^2 the order of s_2 is either 2 or 4. In the former case $\{s_1, s_2\}$ is the tetrahedral group. In the latter case, $\{s_2, s_1^{-1} s_2 s_1, s_1^{-2} s_2 s_1^2\}$ is the quaternion group. As this is invariant under $\{s_1, s_2\}$ we have the result: *If the two generators of a group satisfy the conditions* $s_1^3 = s_2^2$, $(s_1^{-1} s_2)^3 = 1$, *it is either the tetrahedral group or the group of order* 24 *which does not contain a subgroup of the order* 12.

§ 2. *Generalization of the octahedral group.*

The octahedral group is completely defined by the fact that it may be generated by two operators of orders 2 and 4, respectively, such that their product is of order 3. That is, s_1, s_2 define the octahedral group provided they satisfy the equations $s_1^2 = s_2^4 = 1$, $(s_1 s_2)^3 = 1$, and the orders of s_1, s_2 are 2 and 4 respectively. We shall now consider the generalization of the octahedral group given by the equations $s_1^2 = s_2^4$, $(s_1 s_2)^3 = 1$.

It will be convenient to consider the three conjugates of s_2^2 under $s_1 s_2$. These are

$$s_2^2, \qquad s_2^{-1} s_1^{-1} s_2^2 s_1 s_2, \qquad s_1 s_2 s_2^2 s_2^{-1} s_1^{-1} = s_1 s_2^2 s_1^{-1}.$$

The form of the last one of these three conjugates results from the fact that $(s_1 s_2)^{-2} = s_1 s_2$ since $(s_1 s_2)^3 = 1$. These conjugates have a common square since s_2^4 is invariant under $\{s_1, s_2\}$. The group generated by any two of them is therefore a generalized dihedral rotation group. We shall consider the group

*Bulletin of the American Mathematical Society, vol. 1 (1897), p. 218; cf. MANNING, Transactions of the American Mathematical Society, vol. 7 (1906), p. 233.

†Transactions of the American Mathematical Society, vol. 1 (1900), p. 67.

generated by the last two. The product of the third into the inverse of the second is $s_1 s_2 s_2^2 s_1 s_2 s_2^{-2} s_1 s_2 = s_1 s_2^{-1} s_1 s_2^{-1} s_1 s_2^5 = s_2 s_1^5 s_1 s_2^5 = s_2^{18}$; for, from the given conditions we have $(s_2^{-1} s_1^{-1})^3 = (s_1^{-1} s_2^{-1})^3 = 1$, and therefore

$$(s_1 s_2^{-1})^3 = (s_1^2 s_1^{-1} s_2^{-1})^3 = s_1^6 \qquad \text{or} \qquad (s_1 s_2^{-1})^2 = s_2 s_1^5.$$

Since $s_1 s_2^2 s_1^{-1}$ transforms s_2^{18} into its inverse and has a common square with s_2^2, the order of s_2 is a divisor of 72. The group generated by $s_2^{-1} s_1^{-1} s_2^2 s_1 s_2$ and $s_1 s_2^2 s_1^{-1}$ contains s_2^2 since it contains both s_2^4 and s_2^{18}. This group is transformed into itself by s_2, since this operator transforms the latter generator into the former, as $s_1 s_2^2 s_1^{-1} = s_1^{-1} s_2^2 s_1$, and also transforms their product into itself. Hence it is also transformed into itself by s_1, as it is transformed into itself by $s_1 s_2$. That is, s_2^2, $s_2^{-1} s_1^{-1} s_2^2 s_1 s_2$, $s_1 s_2^2 s_1^{-1}$ generate a subgroup which is invariant under $\{s_1, s_2\}$. When this subgroup is extended by means of $s_1 s_2$ there results another invariant subgroup since $s_2 s_1 = (s_1^{-1} s_2^{-1})^{-1}$, $s_1 s_2 = s_1^2 s_1^{-1} s_2^{-1} s_2^2$, and s_1^2, s_2^2 are in the preceding invariant subgroup. The order of $\{s_1, s_2\}$ must therefore divide the product obtained by multiplying the order of $\{s_2^{-1} s_1^{-1} s_2^2 s_1 s_2, s_1 s_2^2 s_1^{-1}\}$ by 6. That is, this order divides $6 \cdot 4 \cdot 18 = 432$.

If the order of s_2 is 72, the group generated by s_2^2, $s_2^{-1} s_1^{-1} s_2^2 s_1 s_2$, $s_1 s_2^2 s_1^{-1}$ is of order 72, and its invariant operators constitute the cyclic group of order 18. It may be defined by the fact that it is non-abelian and contains more than one cyclic subgroup of order 36. Its three cyclic subgroups of order 36 are conjugate under $s_1 s_2$ and its cyclic subgroup of order 18 is composed of the invariant operators of $\{s_1, s_2\}$. By extending this group of order 72 by means of $s_1 s_2$ there results the direct product of the cyclic group of order 9 and the group of order 24 which does not contain a subgroup of order 12. As this direct product contains just three subgroups of order 24, one of them is transformed into itself by s_2 and includes s_2^{18}. This subgroup and s_2^9 generate a group of order 48, known* as G_{52}, and hence $\{s_1, s_2\}$ is the direct product of the latter and the cyclic group of order 9. That is, *if* s_1, s_2 *satisfy the conditions* $s_1^2 = s_2^4$, $(s_1 s_2)^3 = 1$, *then either they generate the direct product of the cyclic group of order* 9 *and the group of order* 48 *known as* G_{52} *or they generate a quotient group of this direct product.*

If the order of s_2 is 36, the three operators s_2^2, $s_2^{-1} s_1^{-1} s_2^2 s_1 s_2$, $s_1 s_2^2 s_1^{-1}$ generate a group of order 36. As this is abelian and contains three cyclic subgroups of order 18 it is the direct product of the four-group and the cyclic group of order 9. By extending it by means of $s_1 s_2$ we obtain the direct product of the alternating group of order 12 and the cyclic group of order 9. Hence $\{s_1, s_2\}$ is the direct product of the symmetric group of order 24 and the cyclic group of order 9 whenever the order of s_2 is 36. If the order of s_2 were not divisible by 4,

---

*Quarterly Journal of Mathematics, vol. 30 (1898), p. 258.

s_2^4 would be equal to the square of a generator of s_2 and hence we would be dealing with a group generated by two operators having a common square. Hence it is only necessary to consider the cases when the order of s_2 has one of the following values: 4, 12, 8, 24. In the first two cases, we arrive at the symmetric group of order 24, or the direct product of this and the group of order 3. In the last two cases we arrive at the group of order 48 considered in the preceding paragraph or the direct product of this and the group of order 3. As the considerations are similar to those given above, it seems unnecessary to enter into details, especially since the quotient groups mentioned at the end of the preceding paragraph are so well known from the fact that G_{52} has a $(2, 1)$ isomorphism with the symmetric group of order 24.

A similar generalization of the definition of the octahedral group is given by the equations

$$s_1^2 = s_2^3, \qquad (s_1 s_2)^4 = 1.$$

To find an upper limit for the order of s_2 we may consider the following product of two conjugate operators of order 2:

$$(s_2 s_1 s_2 s_1 \cdot s_1 s_2 s_1 s_2)^2 = (s_1^2 s_2 s_1 s_2^2 s_1 s_2)^2 = s_1^4 s_2 s_1 s_2^2 s_1 s_2^2 s_1 s_2^2 s_1 s_2^2 s_2^{-1} = s_2^{30}.$$

As s_2^{30} is both commutative with $(s_2 s_1)^2$ and transformed into its inverse by this operator, its order cannot exceed 2 and hence the order of s_2 divides 60. We shall first consider the group generated by s_1^5, s_2^5 when the order of s is 60. It is clear that these two operators satisfy the given conditions since

$$(s_1^5 s_2^5)^4 = (s_1^4 s_2^3 s_1 s_2^2)^4 = s_1^{16} s_2^{12} (s_1 s_2^2)^4 = s_2^{60}.$$

It will appear that $\{s_1, s_2\}$ is the direct product of $\{s_1^5, s_2^5\}$ and the cyclic group generated by s_2^{12}. From the equation given above it follows that $\{s_2 s_1 s_2 s_1, s_1 s_2 s_1 s_2\}$ is the octic group. Extending this by means of s_2^{15}, we obtain a group of order 16 which is invariant under $\{s_1, s_2\}$ since s_2 transforms one generator of this octic group into the other, and the product of the two into $s_1^3 s_2^2 s_1 s_2^2 = (s_2 s_1 s_2 s_1)^{-1} \cdot s_2^{15}$. By extending this group of order 16 by means of s_2^5, we get a group of order 48 which is invariant under $\{s_1, s_2\}$ since $s_1^{-1} s_2^5 s_1 = (s_2 s_1 s_2 s_1)^{-1} \cdot s_2^{10}$. Finally, if we extend this by means of s_1^5 we obtain the group of order 96 in question. It is easy to verify that this group may be generated by the following substitutions:

$$s_1 = aa'gg'ee'cc' \cdot bo'hm'fk'di' \cdot id'ob'mh'kf' \cdot jn'pl'nj'lp',$$

$$s_2 = amlgkjeipcon \cdot bdfh \cdot a'm'l'g'k'j'e'i'p'c'o'n' \cdot b'd'f'h'.$$

When the order of s_2 is 60 the invariant subgroup generated by s_2^{12} has only the identity in common with the G_{96} of the preceding paragraph. *Hence*

$\{s_2^{12}, s_1^5, s_2^5\} \equiv \{s_1, s_2\}$ *is the direct product of* G_{96} *and the cyclic group of order* 5. When the order of s_2 is 12 it follows directly from the preceding paragraph that $\{s_1, s_2\}$ is G_{96}. Hence it remains to consider the cases when the order of s_2 is not divisible by 4. If this order is 30 we again consider $\{s_1^5, s_2^5\}$. Hence $\{s_2 s_1 s_2 s_1, s_1 s_2 s_1 s_2\}$ is either the four-group or the group of order 2. In the former case it does not contain s_2^{15}, since one of its two generators is transformed into the other by s_2 and their product is transformed into itself multiplied by s_2^{15} by the same operator. Hence this four-group and s_2^{15} generate the group of order 8 which contains 7 operators of order 2. Just as in the preceding paragraph, we observe that this group of order 8 is invariant under $\{s_1, s_2\}$.

As s_2^5 transforms this group of order 8 into itself and is commutative with only one of its operators of order 2, it transforms one of its subgroups of order 4 into itself and this subgroup and s_2^{10} generate the alternating group of order 12.* That is, s_2^5 and this group of order 8 generate the direct product of the alternating group of order 12 and an operator of order 2. By extending this group of order 24 by s_1^5 we obtain the group of order 48 which has 24 additional operators of order 4 of which 12 have the same square as s_1^5 while the other 12 have a different common square. This elementary group of order 48 is $\{s_1^5, s_2^5\}$ and has the symmetric group of order 24 for its group of cogredient isomorphisms. When s_2 is of order 6, $\{s_1, s_2\}$ is this group of order 48, and when s_2 is of order 30 it is the direct product of this group and the cyclic group of order 5.

It remains only to consider the cases when s_2 is of order 3 or 15. In the former case $\{s_1, s_2\}$ is clearly the octahedron group, while in the latter it is the direct product of this and the cyclic group of order 5, since s_1^5, s_2^5 generate the octahedron group. The proof of this fact follows directly from the preceding paragraphs. *Hence there are just six groups whose two generators satisfy the conditions* $s_1^2 = s_2^3$, $(s_1 s_2)^4 = 1$. *Three of these are of orders* 24, 48 *and* 96, *respectively; while the other three are the direct products of these groups and the cyclic group of order* 5.

It remains to consider the generalization of the octahedral group whose two generators satisfy the equations

$$s_1^3 = s_2^4, \qquad (s_1 s_2)^2 = 1.$$

The superior limit of the order of s_2 may be found as follows:

$$(s_2^{-2} s_1^{-2} s_2^2 s_1^2)^2 = s_2^{-16}(s_2^2 s_1 s_2^2 s_1^2)^2 = s_2^{-24}(s_2 s_1^2)^4 = s_2^{-28},$$

since

$$s_1 s_2^2 = s_2^{-1} s_1^{-1} s_2, \qquad (s_2 s_1^2)^n = s_1^{-1} s_2^{-n} s_1.$$

*Cf. MOORE, Bulletin of the American Mathematical Society, vol. 1 (1894), p. 61.

As s_2^{-2} and $s_1^{-2}s_2^2s_1^2$ have a common square, s_2^{-28} is transformed into its inverse and into itself by s_2^2. That is, the order of s_2 divides 56. The two operators s_1^{49}, s_2^{49} also satisfy the given conditions since $(s_1^{49}s_2^{49})^2 = s_2^{224}(s_1s_2)^2 = 1$. If the order of s_2 is 56 the orders of these two operators are 6 and 8 respectively. We proceed to consider the group generated by them.

From the given equations it follows that s_2^{98}, $s_1^{-98}s_2^{98}s_1^{98}$ generate the quaternion group. That this is invariant under $\{s_1, s_2\}$ follows from the proof that $\{s_2^2, s_1^{-2}s_2^2s_1^2\}$ is invariant under $\{s_1, s_2\}$. That this group contains

$$s_1^{-1}s_2^2s_1 = s_2^{-4}\cdot s_1^2s_2^2s_1 = s_2^{-4}s_1s_2^{-1}s_1^{-1}s_2s_1 = s_2^{-4}s_1s_2^{-1}s_1^{-2}s_2^{-1} = s_2^{-8}s_1s_2^{-2}s_1^{-1}s_2^{-2} = s_2^{-20}s_1s_2^2s_1^2s_2^2$$

follows from the given form of the product $s_2^{-2}\cdot s_1^{-2}s_2^2s_1^2$. It is transformed into itself by s_2 since s^2 transforms $s_2^2s_1s_2^2s_1^2$ into $s_2s_1s_2^2s_1^2s_2 = s_1^3\cdot s_1^{-1}s_2^2s_1s_2^2$, which is in this group. The quaternion group $\{s_2^{98}, s_1^{-98}s_2^{98}s_1^{98}\}$ and s_1^{49} generate the group of order 24 which does not contain a subgroup of order 12. As this is clearly invariant under $\{s_1, s_2\}$ it results that $\{s_1^{49}, s_2^{49}\}$ is the group of order 48 known* as G_{51}, which is very closely related to the group of this order G_{52} considered at the beginning of this section. When the order of s_2 is 8, $\{s_1, s_2\}$ is therefore G_{51}, and when this order is 56, $\{s_1, s_2\}$ is the direct product of G_{51} and the cyclic group of order 7, since the group generated by s_2^8 is invariant under $\{s_1^{49}, s_2^{49}\}$ and has only the identity in common with it.

When the order of s_2 is 28, $\{s_1^{49}, s_2^{49}\}$ is the symmetric group of order 24 and $\{s_1, s_2\}$ is the direct product of it and the cyclic group of order 7. Hence *there are exactly four groups whose two generating operators satisfy the conditions* $s_1^3 = s_2^4$, $(s_1s_2)^2 = 1$. *Two of these are of orders* 24 *and* 48 *respectively while the other two are the direct products of these and the cyclic group of order* 7. It may be added that the three groups of order 48 which have been considered in this section are distinct.

§ 3. *Generalization of the icosahedral group.*

If two operators s_1, s_2 satisfy the equations

$$s_1^2 = 1, \qquad s_2^5 = 1, \qquad (s_1s_2)^3 = 1,$$

they generate the simple group of order 60. This fact was observed by HAMILTON as early as 1856 †, and it is of especial interest since there are only a few finite groups which may be defined in such an elementary manner. It is well known that the 60 operators of this group may be written as follows:

$$s_2^m,\ s_2^m s_1 s_2^n,\ s_2^m s_1 s_2^2 s_1 s_2^n,\ s_2^m s_1 s_2^2 s_1 s_2^3 s_1 \quad (m, n = 1, 2, \cdots, 5).$$

*Quarterly Journal of Mathematics, vol. 30 (1898), p. 258.

† HAMILTON, Philosophical Magazine, vol. 12 (1856), p. 446.

In the following generalization it will be convenient to use a group which is defined by a very closely related set of equations, viz.,

$$s_1^4 = 1, \qquad s_2^5 = s_1^2, \qquad (s_1 s_2)^3 = 1.$$

Since the icosahedral group does not involve any operators of order 4 the order of the present group is not less than 120. That it is exactly of this order follows from the fact that each of its operators may be written in one of the following forms:

$$s_2^m,\ s_2^m s_1 s_2^n,\ s_2^m s_1 s_2^2 s_1 s_2^n,\ s_2^m s_1 s_2^2 s_1 s_2^3 s_1 \quad (m=1, 2, \cdots, 10;\ n=1, 2, \cdots, 5).$$

The reductions to these forms follow directly from those of the icosahedral group, since any product involving powers of s_1 and s_2 may be written as a power of s_2 multiplied by a product in which only the first power of s_1 and the first four powers of s_3 occur; for s_1^2 and s_2^5 are invariant under $\{s_1, s_2\}$.

This group (G_{120}) of order 120 is well known as an imprimitive group of degree 24 * and it seems therefore unnecessary to give an existence proof here. It may be observed that it does not include the icosahedral group, but this group is its group of cogredient isomorphisms. It will be found that the groups whose two generating operators satisfy the generalization of the icosahedral group given by the equations

$$s_1^2 = s_2^5, \qquad (s_1 s_2)^3 = 1$$

are merely direct products of a cyclic group whose order divides 21 and one of the two groups given above. As a first step in this proof, we consider the product of two operators of order two which have a common square, viz., s_1 and $s_2^{-4} s_1 s_2^4$. The following equations result directly from the two given relations between s_1 and s_2:

$$(s_2^{-4} s_1 s_2^4 s_1^{-1})^2 = (s_2^2 s_1^{-1} s_2^{11})^2 = s_2^2 s_1^{-1} s_2^{13} s_1^{-1} s_2^{11},$$

$$(s_2^{-4} s_1 s_2^4 s_1^{-1})^5 = s_2^2 (s_1^{-1} s_2^{13})^5 s_2^{-2} = s_2^2 (s_1 s_2^3)^5 s_2^{23} = s_2^{105}.$$

Since s_1 transforms s_2^{105} into its inverse † and is commutative with it, the order of s_2 cannot exceed 210. If this order is 210 the two operators s_1^{21}, s_2^{21} generate the given group of order 120 since $(s_1^{21} s_2^{21})^3 = s_2^{210}(s_1 s_2)^3 = 1$. Hence $\{s_1, s_2\}$ contains the group generated by s_2^{10} and this group of order 120 as invariant subgroups having only the identity in common. It is therefore the direct product of these two subgroups, and each of its operators is represented once and only once by one of the following expressions:

$$s_2^m,\ s_2^m s_1 s_2^n,\ s_2^m s_1 s_2^2 s_1 s_2^n,\ s_2^m s_1 s_2^2 s_1 s_2^3 s_1 \quad (m=1, 2, \cdots, 210;\ n=1, 2, \cdots, 5).$$

* It is also the compound perfect group of lowest possible order. Cf. American Journal of Mathematics, vol. 20 (1898), p. 277.

† Archiv der Mathematik und Physik, vol. 9 (1895), p. 6.

The cyclic group generated by s_2^5 is composed of the invariant operators of this group and the corresponding quotient group is the icosahedral group. This group of order 2520 may be regarded as the generalization of the icosahedral group which is defined by the equations $s_1^2 = s_2^5$ and $(s_1 s_2)^3 = 1$. The other groups whose two generators satisfy these conditions must also satisfy an additional condition, viz., that the order of s_2 is less than 210. If the order of s_2 is either 70 or 30 it still follows that $(s_2^{21} s_2^{21})^3 = 1$ and hence the corresponding $\{s_1, s_2\}$ is the direct product of the given group of order 120 and either the cyclic group of order 7 or the cyclic group of order 3.

Finally when the order of s_2 is 105, 35 or 15, the two operators s_1^{21}, s_2^{21} generate the icosahedral group and $\{s_1, s_2\}$ is the direct product of this group and the cyclic groups of orders 21, 7, 3, respectively. If we include the icosahedral group and the given group of order 120, *there are exactly eight groups whose two generators satisfy the equations* $s_1^2 = s_2^5$, $(s_1 s_2)^3 = 1$. In each case the operator of the group is represented once and only once by one of the following forms:

$$s_2^m, \quad s_2^m s_1 s_2^n, \quad s_2^m s_1 s_2^2 s_1 s_2^n, \quad s_2^m s_1 s_2^2 s_1 s_2^3 s_1 \qquad (m = 1, 2, \cdots, \text{order of } s_2;\ n = 1, 2, \cdots, 5).$$

It may be added that the existence of the given direct products could have been seen directly, since, in the definition of the icosahedron group by means of the equations $s_1^2 = 1$, $s_2^5 = 1$, $(s_1 s_2)^3 = 1$, it is clearly possible to select an independent operator of order 21 and multiply s_1, s_2, respectively, by its first and its thirteenth powers. The resulting products will satisfy the equations of the generalized icosahedral group. The above proof may therefore be regarded as merely establishing that there are no other such groups.

The two other possible similar generalizations of the icosahedron group can readily be reduced to the preceding one. We shall first consider the case when

$$s_1^3 = s_2^5, \qquad (s_1 s_2)^2 = 1.$$

To exhibit the connection with the preceding case we let $t_1 = s_1 s_2$ and $t_2 = s_2^{-1}$. Hence we have to consider $\{t_1, t_2\}$ where

$$t_1^2 = 1, \qquad t_2^5 = (t_1 t_2)^{-3}.$$

To obtain an upper limit for the order of t_2 we may use the product $t_2^{-4} t_1 t_2^4 t_1$. Since t_2^5 is invariant under $\{t_1, t_2\} \equiv \{s_1, s_2\}$, it is possible to write the given product in a more compact form as follows: From $t_2^{-1} t_1 t_2^{-1} t_1 t_2^{-1} t_1 = t_2^5$ it results that $t_2^{-4} t_1 t_2^4 t_1 = t_2^2 t_1 t_2 \cdot t_2^5$. Hence it results that $(t_2^{-4} t_1 t_2^4 t_1)^n = t_2^2 (t_1 t_2^3)^n t_2^{5n-2}$. If we write the equation $t_2^{-1} t_1 t_2^{-1} t_1 t_2^{-1} t_1 = t_2^5$ in the form $t_1 t_2^3 = t_2 t_1 t_2 t_1 t_2^4 \cdot t_2^5$ it is clear that $(t_1 t_2^3)^n = t_2 t_1 t_2^n t_1 t_2^{-1} \cdot t^{10n}$. Letting $n = 5$ in these equations we arrive at the equation

$$(t_2^{-4} t_1 t_2^4 t_1)^5 = t_2^{80}.$$

As t_1 is commutative with t_2^{80} and also transforms it into its inverse, the order of t_2 is a divisor of 160. We shall now prove that it cannot be 160. If t_2 were of order 160 $t_1 t_2$ would be of order 96. The group generated by t_2^{16} and $(t_1 t_2)^{16} = t_1 t_2 \cdot t_2^{-25}$ is the same as the one generated by t_2^{16} and $t_1 t_2 \cdot t_2^{-25} \cdot t_2^{80} = t_1 t_2 \cdot t_2^{55}$. As the last operator is of order 3 and as $t_1 t_2^{56} \cdot t_2^{-16} = t_1 t_2^{40}$ is of order 4 we are dealing with a group whose two generators s_1, s_2 satisfy the equations $s_1^2 = s_2^5$, $(s_1 s_2)^3 = 1$, $s_1^4 = 1$. This is the group of order 120 whose properties were considered above. The cyclic group generated by t_2^{15} is invariant under this group and with it generates $\{t_1, t_2\}$. This is impossible, since t_1 would correspond to an operator of order 4 in the $(k, 1)$ isomorphism between $\{t_1, t_2\}$ and the given group of order 120. The order of s_2 must therefore divide 80.

It is now easy to prove that *there are exactly five groups whose two generators* s_1, s_2 *satisfy the equations* $s_1^3 = s_2^5$, $(s_1 s_2)^2 = 1$. *They are the direct products of the icosahedron group and the cyclic group of order* 2^α, $\alpha = 0, 1, 2, 3, 4$. The proof of this theorem follows very readily from the fact that the order of s_2 divides 80. Hence $\{s_1^{16}, s_2^{16}\}$ is the icosahedron group, since $(s_1^{16} s_2^{16})^2 = (s_1 s_2)^2 = 1$ and s_1^{16}, s_2^{16} are of orders 3 and 5 respectively. As $\{s_1, s_2\}$ contains the cyclic group generated by s_2^5 and the icosahedron group as invariant subgroups having only the identity in common and the order of $\{s_1, s_2\}$ cannot exceed the product of the orders of this invariant subgroup, the proof is complete.

It remains only to consider the groups whose two generators satisfy the equations

$$s_1^2 = s_2^3, \qquad (s_1 s_2)^5 = 1.$$

It will be convenient to use the auxiliary symbols $t_1 = s_1$, $t_2 = s_1 s_2$. Proceeding as before we have

$$(t_2^{-4} t_1 t_2^4 t_1^{-1})^n = t_2^2 (t_1 t_2^3)^n t_2^3 s_2^{-12n}, \qquad (t_1 t_2^3)^n = t_2 t_1 t_2^n t_1 t_2^4 \cdot s_2^{-6n-3}.$$

Hence

$$(t_2^{-4} t_1 t_2^4 t_1^{-1})^5 = s_2^{-90}.$$

Moreover, $(t_2^{-2} t_1 t_2^2 t_1^{-1})^3 = s_2^{42} (t_2^2 t_1)^5 t_2^{-8} = s_2^{75}$. Since t_1 transforms both s_2^{-90} and s_2^{75} into their inverses and is commutative with each of them, the order of s_2 divides 30.

If the order of s_2 is 30 that of s_1 is 20. In this case s_1^5, s_2^{10} generate the group G_{120} considered above; for $(s_1^5 s_2^{10})^5 = s_2^{15}$ and the orders of s_1^5, s_2^{10} are 4 and 3 respectively. As s_1^4 is invariant under this group and has only the identity in common with it and as $\{s_1^4, s_1^5, s_2^{10}\} \equiv \{s_1, s_2\}$ it follows that the latter is the direct product of G_{120} and the cyclic group of order 5. When the order of s_2 is 15 that of s_1 is 10 and s_1^5, s_2^{10} generate the icosahedron group. In this case $s_1^5 s_2^{10} = s_2^{15} s_1 s_2 = s_1 s_2$. When the order of s_2 is either 6 or 3, $\{s_1, s_2\}$ is either the given group of order 120 or the icosahedron group. Hence the theorem:

There are exactly four groups whose two generators s_1, s_2 *satisfy the equations* $s_1^2 = s_2^3$, $(s_1 s_2)^5 = 1$. *They are the icosahedron group,* G_{120}, *and the direct products of these groups and the cyclic group of order* 5.

Combining the results of this section, we conclude that there are fourteen distinct groups which are generated by two operators satisfying one of the three immediate generalizations of the defining relations of the icosahedron group. The generators of the icosahedron group satisfy each of these generalized definitions, those of G_{120} satisfy two of them, while those of each of the other twelve groups determined above satisfy only one of these sets of conditions. The simple defining relations and the close contact with the simple group of lowest composite order make these fourteen groups of peculiar interest.